Waves in periodic media:
edge effects and topology

Master “Wave Physics & Acoustics”
and “Recherche Acoustique” (2024)

Laboratoire de Mécanique et d'Acoustique

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Outline

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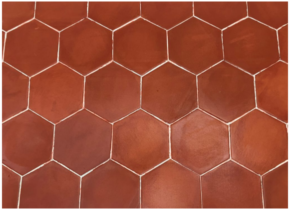
1. Introduction
   - Context: periodic media
2. Periodic media
   - Example: Double chain of masses and springs
   - General theory: Bloch-Floquet method
   - Introducing edges: Finite systems
3. Topology of eigenvalue problems
   - Simple example: real 2×2 matrices
   - Berry connection
   - Zak phase
4. Bulk-boundary correspondance
   - Continuous 1D systems: topological interface modes
   - 2D systems: breaking reciprocity
   - Reciprocal 2D systems: the valley Hall effect

Foot steps

Knitting pattern

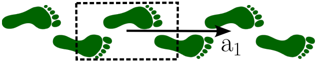


Typical tilling in Marseille: “tomettes”

What is a periodic medium?

Repeating patterns

Context: periodic media



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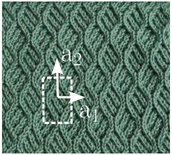

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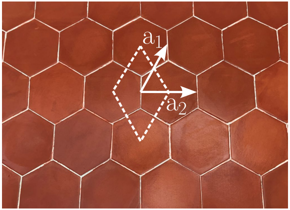


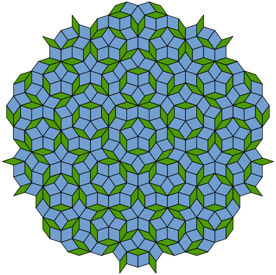
Typical tilling in Marseille: “tomettes”

More precisely:

- Find a elementary pattern: **a unit cell**
- Periodic: full space obtained by integer translations of vectors

$$\mathrm{a}_1, \mathrm{a}_2, \ldots \mathrm{a}_\mathrm{d}$$

with $d = 1, 2$ or 3, the dimension of space



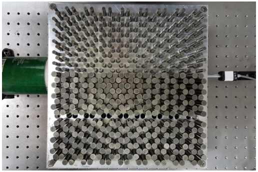
Penrose tiling: quasi-periodic

Granular: random medium

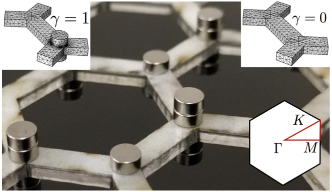
Different from:

- Quasi-periodic medium
- Random medium

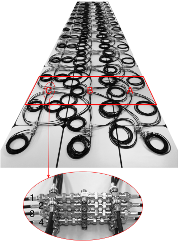
Context: periodic media



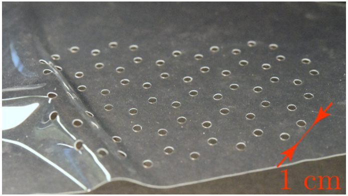
Acoustic waves



Elastic waves



Optical waves



Hydro-elastic waves

Everything we discuss valid for many types of waves!

Outline

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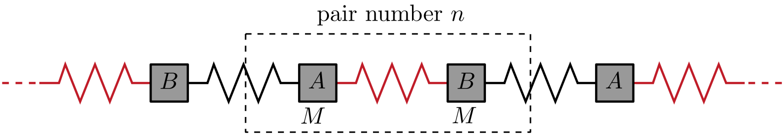
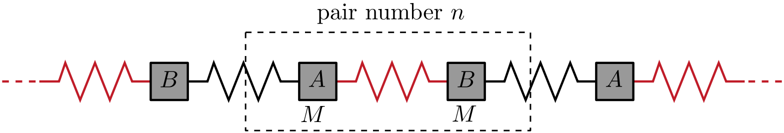
Chain of masses with 2 spring constants



- Label each **pair of masses** with integer $n \in \mathbb{Z}$
- Newton's second law:

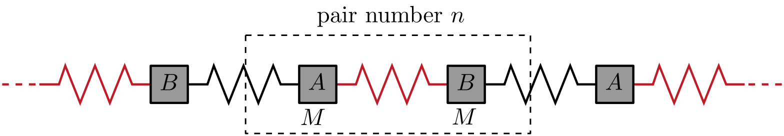
$$\begin{aligned} -M\ddot{x}_n^A &= -\kappa_0 \left(x_n^A - x_n^B\right) - \kappa_1 \left(x_n^A - x_{n-1}^B\right) \\ -M\ddot{x}_n^B &= -\kappa_0 \left(x_n^B - x_n^A\right) - \kappa_1 \left(x_n^B - x_{n+1}^A\right) \end{aligned}$$

How is wave propagation in this system?



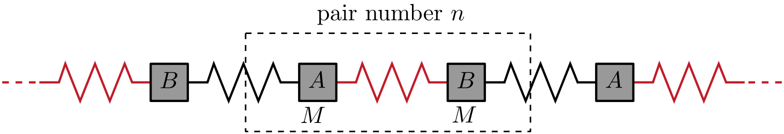
- Fixed frequency ω
- Complex displacement amplitudes:

$$\begin{aligned} x_n^A(t) &= \mathrm{Re}\left(A_n e^{-i\omega t}\right) \\ x_n^B(t) &= \mathrm{Re}\left(B_n e^{-i\omega t}\right) \end{aligned}$$



- Newton's second law:

$$\begin{aligned} -\omega^2 M A_n &= -\kappa_0 \left(A_n - B_n\right) - \kappa_1 \left(A_n - B_{n-1}\right) \\ -\omega^2 M B_n &= -\kappa_0 \left(B_n - A_n\right) - \kappa_1 \left(B_n - A_{n+1}\right) \end{aligned}$$



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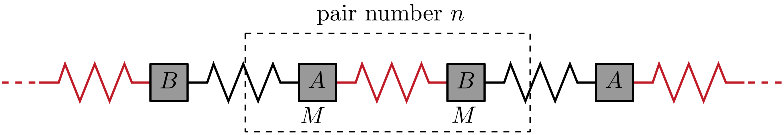
- **Bloch condition:** Fixed q, impose:

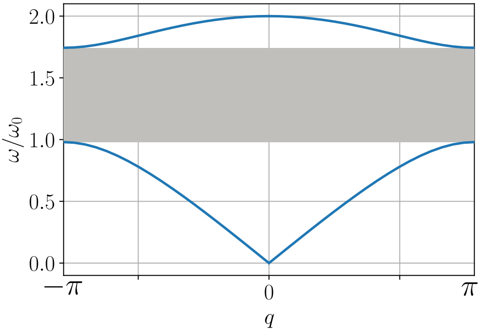
$$\begin{aligned} A_{n+1} &= e^{iq} A_n \\ B_{n+1} &= e^{iq} B_n \end{aligned}$$



- Equation becomes an **eigenvalue problem:**

$$\omega^2 \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = \frac{1}{M} \begin{pmatrix} \kappa_0 + \kappa_1 & -\kappa_0 - \kappa_1 e^{-iq} \\ -\kappa_0 - \kappa_1 e^{iq} & \kappa_0 + \kappa_1 \end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}$$

- Eigenvalues $\omega_1(q)$, $\omega_2(q)$



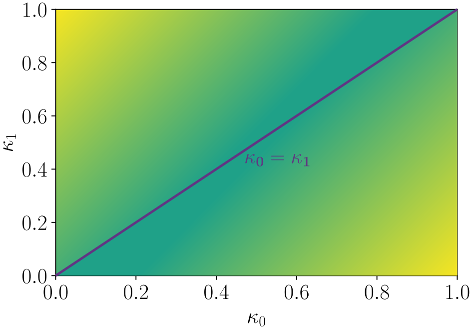
With:

$$\omega_0 = \sqrt{\frac{\kappa_0 + \kappa_1}{2M}}$$

- Dispersion relation with 2 bands:

$$\omega_1^2(q) = \frac{\kappa_0 + \kappa_1}{M} - \frac{\sqrt{\kappa_0^2 + \kappa_1^2 + 2\kappa_0\kappa_1 \cos(q)}}{M}$$
$$\omega_2^2(q) = \frac{\kappa_0 + \kappa_1}{M} + \frac{\sqrt{\kappa_0^2 + \kappa_1^2 + 2\kappa_0\kappa_1 \cos(q)}}{M}$$

- Between: **gap**



- **Parameter space**
- Gap size
- **Gap closes** when $\kappa_0 = \kappa_1$

Outline

General case:

- Eigenvalue problem:

$$\varepsilon\psi = \mathcal{H}\psi$$

- ε: eigenvalue
- $\mathcal{H}$: matrix or linear operator
- ψ: wave function (pressure, displacement, electric field, etc.)

Double mass-spring chain example:

- Eigenvalue $\varepsilon = \omega^2$
- Matrix or linear operator

$$\mathcal{H} = \frac{1}{M}\begin{pmatrix} \ddots & \ddots & & & & \\ \ddots & \kappa_0+\kappa_1 & -\kappa_0 & & & \\ & -\kappa_0 & \kappa_0+\kappa_1 & -\kappa_1 & & \\ & & -\kappa_1 & \kappa_0+\kappa_1 & -\kappa_0 & \\ & & & -\kappa_0 & \kappa_0+\kappa_1 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}$$

- Wave function

$$\psi = \begin{pmatrix} \vdots \\ A_n \\ B_n \\ \vdots \end{pmatrix}$$

A continuous example: Helmholtz equation

$$\Delta p + k^2 p = 0$$

- Eigenvalue $\varepsilon = k^2$
- Matrix or linear operator

$$\mathcal{H} = -\Delta$$

- Wave function

$$\psi = p(x)$$

Bloch-Floquet theorem:

- System invariant under translation $x \to x + a$

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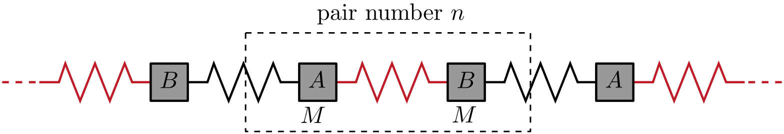
- Set of eigenvalues $\varepsilon_j(q)$ gives **dispersion relation** (bands)



- Translation invariance:

$$
\begin{aligned}
A_n &\rightarrow A_{n+1} \\
B_n &\rightarrow B_{n+1}
\end{aligned}
$$

- **Bloch condition:**

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\begin{aligned}
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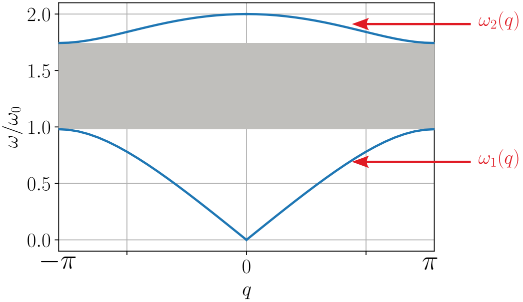

- Dispersion relation $\omega_1(q)$, $\omega_2(q)$

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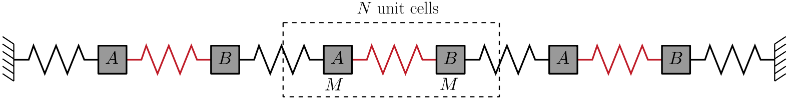
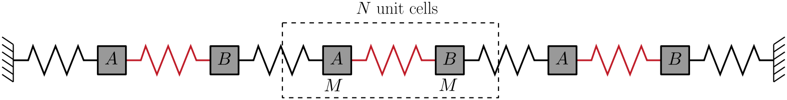


- $N \gg 1$ unit cells
- Hard wall **boundary conditions**
- Finite number of eigenmodes



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Numerical exercice:

$M = 1$, $\kappa_0 = 1.5$, $\kappa_1 = 0.5$, and $N = 10$

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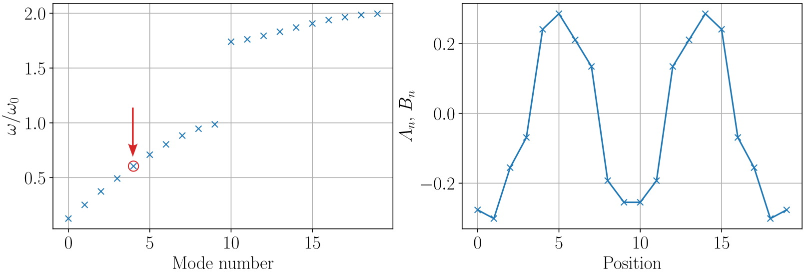

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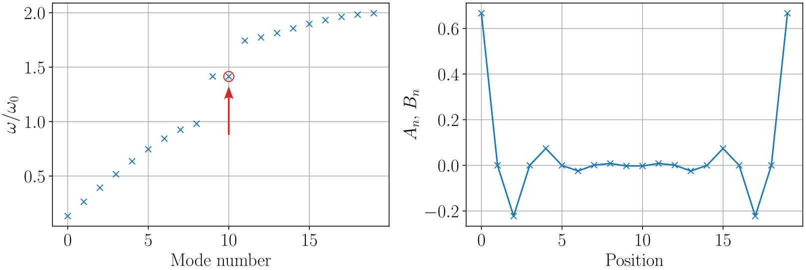

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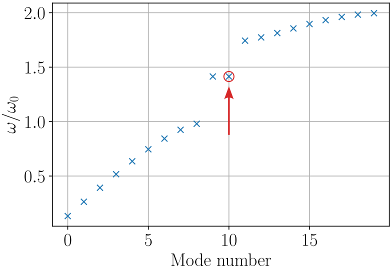

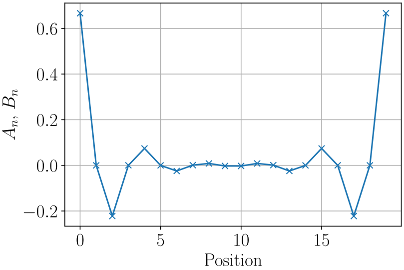

Edge mode with in-gap frequency!

When do you have edge modes?

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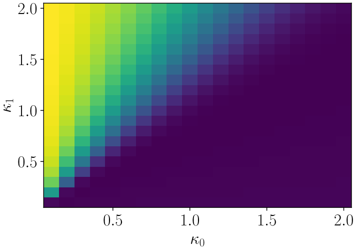
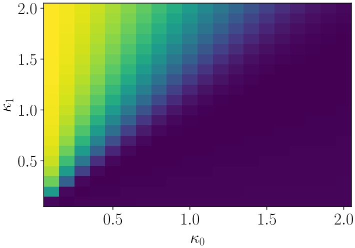


Localization strength

When do you have edge modes?



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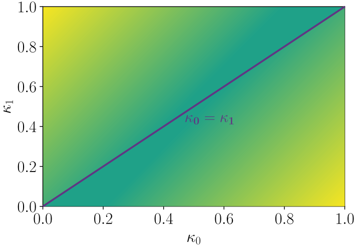
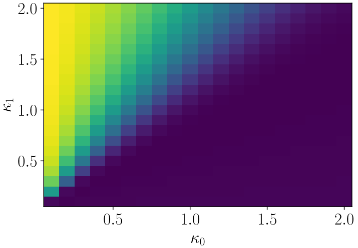


Size of gap

When do you have edge modes?



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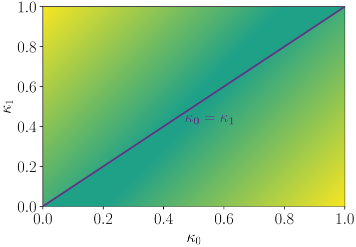

Size of gap

In the limit $N \gg 1$, edge modes are present iff $\kappa_0 < \kappa_1$

Raises questions:

- Two diagrammes always coincide?
- Which “side” has edge modes?

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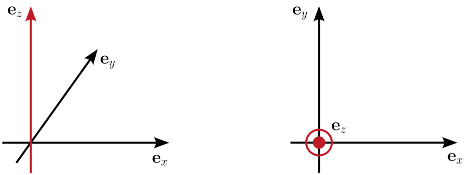


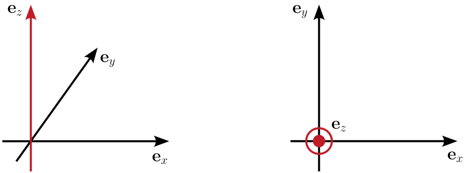
- Real valued 2×2 matrix

$$H = \begin{pmatrix} z + x & -y \\ -y & z - x \end{pmatrix}$$

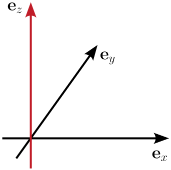
- Parameter space $\mathbb{R}^3 = \{(x, y, z)\}$
- 2 eigenvalues:

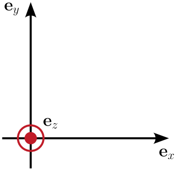
$$\varepsilon_{\pm}(x, y, z) = z \pm \sqrt{x^2 + y^2}$$

- Degeneracy line: z-axis ($x = 0$ and $y = 0$)



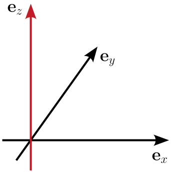
- Eigenvalues are smooth outside of degeneracy line
- What about eigen-vectors?

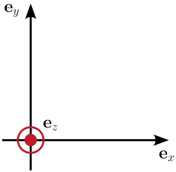




- Eigenvalues are smooth outside of degeneracy line
- What about eigen-vectors?

$$\varphi_-(x,y,z) = \frac{1}{\sqrt{y^2 + (x + \sqrt{x^2+y^2})^2}} \begin{pmatrix} y \\ x + \sqrt{x^2+y^2} \end{pmatrix}$$





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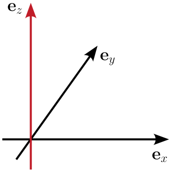
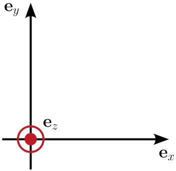
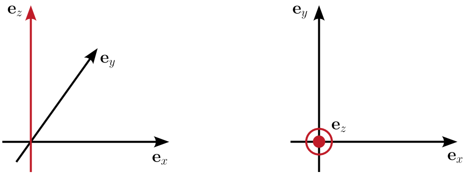
- Real valued but **not** smooth!

$$\lim_{\substack{x<0 \\ y \to 0^\pm}} \varphi_-(x,y,z) = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}$$



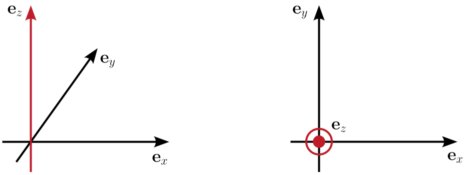


- Easier to see in cylindrical coordinates $(x, y, z) \to (r, \theta, z)$



- Easier to see in cylindrical coordinates $(x, y, z) \to (r, \theta, z)$

$$\varphi_- = \begin{pmatrix} \sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}$$



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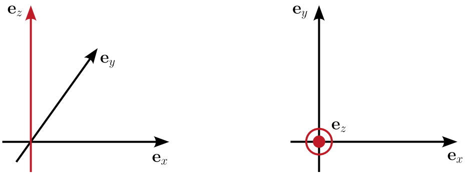
- Should be $\theta \to \theta + 2\pi$ periodic!
- Jump $\lim_{\theta \to \pi} \varphi_- = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ but $\lim_{\theta \to -\pi} \varphi_- = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$



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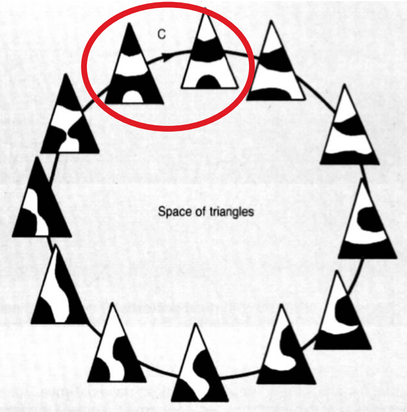
This is called a topological obstruction



- Eigen-modes of a triangle cavity
- Top summit moves around a loop
- After a round trip: factor -1

Back to 2×2 matrices

- Solution to get rid of discontinuity → **complex** values

$$\varphi_- = e^{i\theta/2} \begin{pmatrix} \sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix} = \frac{i}{2} \begin{pmatrix} 1 - e^{i\theta} \\ -i(1 + e^{i\theta}) \end{pmatrix}$$

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Raises more questions:

- Overall phase of φ **not unique**. Is there a prescription?
- When is there a topological obstruction?
- What is independent of the choice of phase?

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Uniqueness and regularity of $\varepsilon(\mathbf{q})$ and $\varphi(\mathbf{q})$?

- Eigenvalue problem with parameters

$$\varepsilon(\mathbf{q})\varphi(\mathbf{q}) = H(\mathbf{q}) \cdot \varphi(\mathbf{q})$$

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- **What about $\varphi(\mathbf{q})$?**

- Ambiguity: if $\alpha(\mathbf{q})$ smooth function

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Is there a "natural" way of choosing the phase?

Short answer: yes and no

- Globally: no
- From one point to the next: yes

- Choose one option for the eigenvector $\varphi(\mathbf{q})$
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- New Berry connection

$$\tilde{A}(\mathbf{q}) = -i\langle\tilde{\varphi}|\nabla_{\mathbf{q}}\tilde{\varphi}\rangle = A(\mathbf{q}) + \nabla_{\mathbf{q}}\alpha$$

defined up to a gradient (like velocity potential, or magnetic potential)

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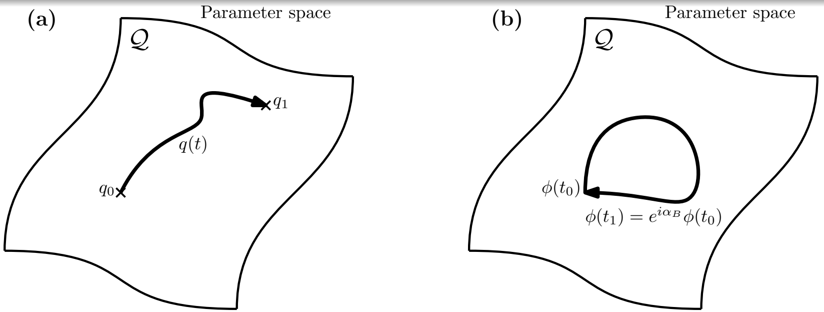
  defined up to a gradient (like velocity potential, or magnetic potential)
- Also **Berry curvature**:

$$F(\mathbf{q}) = \nabla_{\mathbf{q}} \wedge A(\mathbf{q})$$

  is a gauge independent quantity ($\tilde{F} = F$)



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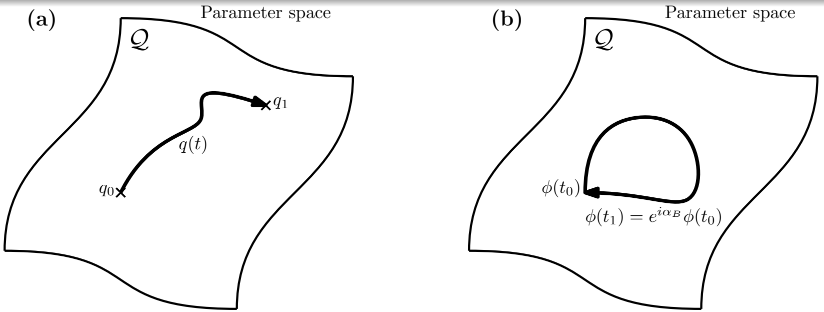
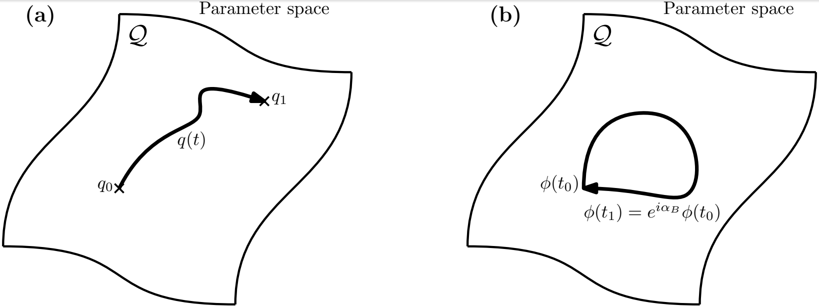


Back to: **Is there a "natural" way of choosing the phase?**

- Along a **path**

$$\phi(t) = e^{-i\int A(\mathbf{q}')\mathrm{d}\mathbf{q}'}\varphi(\mathbf{q}(t))$$

- Parallel transport
- Also adiabatic transport



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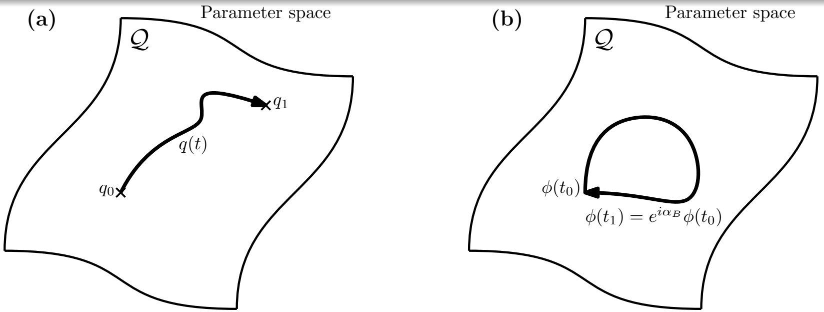
- Along a **loop**: ϕ is not single valued, $\phi(t_1) \neq \phi(t_0)$



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- Along a **loop**: ϕ is not single valued, $\phi(t_1) \neq \phi(t_0)$
- Differ by a phase
- The phase shift is the Berry phase:

$$\alpha_B = \oint A(\mathbf{q}) \mathrm{d}\mathbf{q}$$

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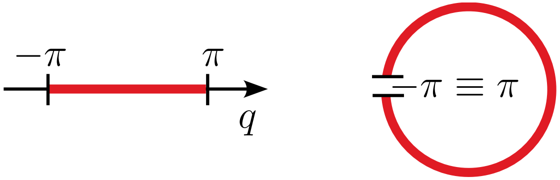
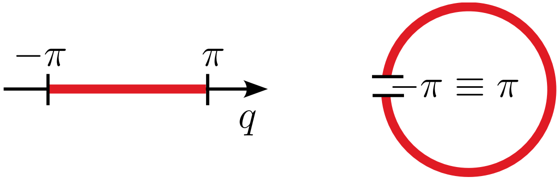


- Back to Bloch-Floquet eigenvalue problem:

$$\varepsilon(q)\varphi(q) = H(q)\varphi(q)$$



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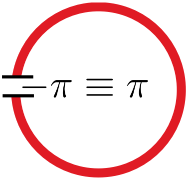

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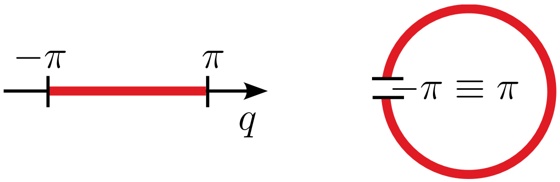

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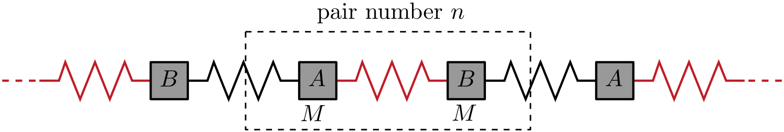
- In periodic systems: parameter is Bloch wavenumber q
- In 1D: q-space $\mathcal{Q}$ is a loop
- **Each band** has a Berry phase, called the **Zak phase**

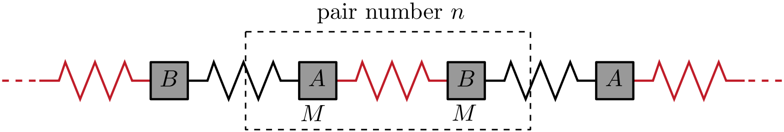
$$\alpha_Z = \int_{-\pi}^{\pi} A_j(q)\mathrm{d}q$$



- Double mass-spring chain example

$$H(q) = \frac{1}{M} \begin{pmatrix} \kappa_0 + \kappa_1 & -\kappa_0 - \kappa_1 e^{-iq} \\ -\kappa_0 - \kappa_1 e^{iq} & \kappa_0 + \kappa_1 \end{pmatrix}$$

- Eigenvalues $\omega_1(q)$, $\omega_2(q)$



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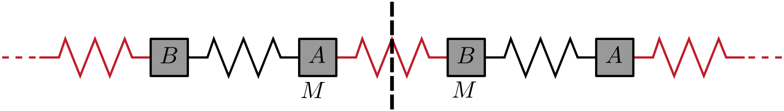
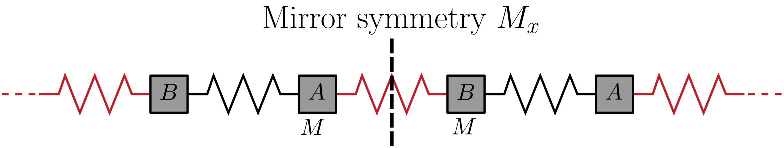
Numerical exercice

Mirror symmetry M_x



- Why is $\alpha_Z \equiv 0$ or $\pi \quad [2\pi]$?



- Why is $\alpha_Z \equiv 0$ or $\pi \quad [2\pi]$?
- Origin: mirror symmetry

$$M_x H(q) M_x = H(-q)$$

Zak phase in mirror symmetric systems

Zak phase in mirror symmetric systems

- Mirror symmetry

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- If $\varphi(q)$ is eigen-vector of $H(q)$, then $M_x \varphi(q)$ is eigen-vector of $H(-q)$ with the same eigen-value

Zak phase in mirror symmetric systems

- Mirror symmetry

$$M_x H(q) = H(-q) M_x$$

- If $\varphi(q)$ is eigen-vector of $H(q)$, then $M_x \varphi(q)$ is eigen-vector of $H(-q)$ with the same eigen-value
- $M_x \varphi(q)$ and $\varphi(-q)$ differ by a phase

$$\varphi(-q) = e^{i\theta(q)} M_x \varphi(q)$$

- Consequence: **Berry connection**

$$-A(-q) = A(q) + \frac{\mathrm{d}\theta}{\mathrm{d}q}$$

- Leads to $\alpha_Z \equiv 0[\pi]$

Zak phase in mirror symmetric systems

Zak phase in mirror symmetric systems

- We can go further: **relate to symmetries**

Zak phase in mirror symmetric systems

- We can go further: **relate to symmetries**
- At $q = 0$ and $q = \pi$ we have $q \equiv -q[2\pi]$
- Hence, at these points

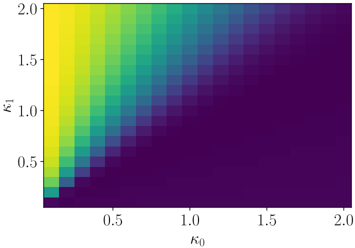
$$\varphi = \pm M_x \varphi$$

- In other words: $\theta(0)$ and $\theta(\pi)$ are 0 or π mod 2π
- α_Z gives the **change of symmetry** of eigenvectors:

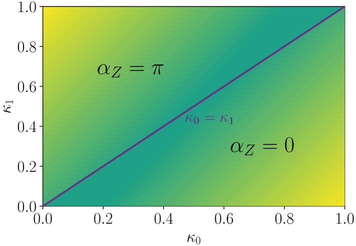
$$\alpha_Z = \theta(0) - \theta(\pi)$$

New phase diagramme

New phase diagramme



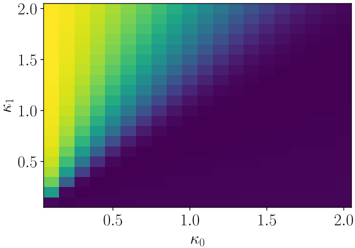
Localization strength



Gap + **topological indices**

Number of modes $= \alpha_Z/\pi$

New phase diagramme



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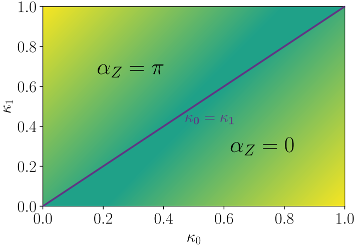

Gap + **topological indices**

Number of modes $= \alpha_Z / \pi$

This is called bulk-boundary correspondance

Outline

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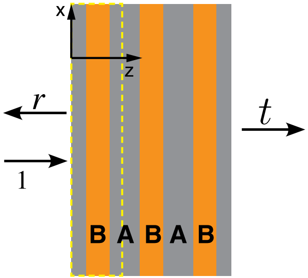
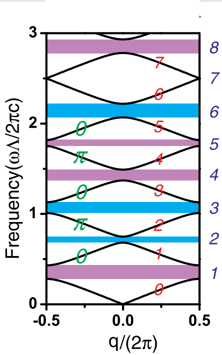
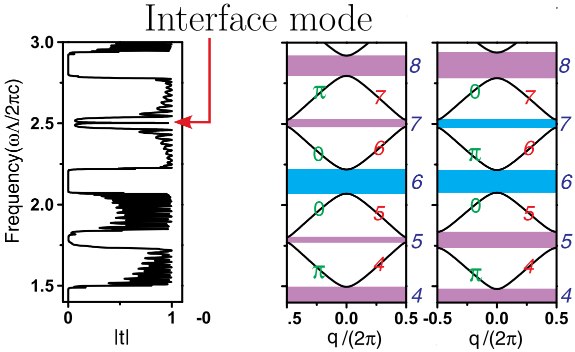




[Xiao, Zhang, Chan, *PRX* 2014]

- Bilayer material
- Topological invariant of a **given gap**:

$$N_n \equiv \frac{1}{\pi} \sum_{j=0}^{n-1} \alpha_Z^{(j)} + n \quad [2]$$



[Xiao, Zhang, Chan, *PRX* 2014]

- In-gap mode if different topological invariants
- Zak phase again predicts presence of localized modes!

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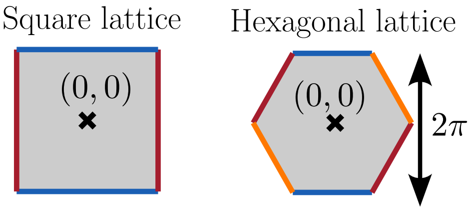


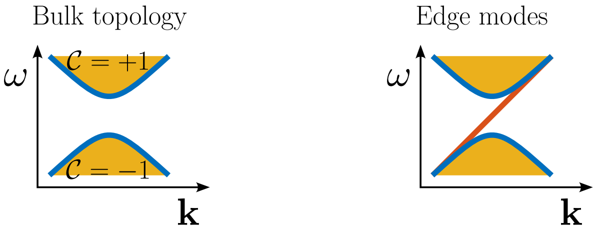
- In 2D: the Brillouin zone is a **torus**
- Berry curvature:

$$F(q_1, q_2) = \nabla \wedge A = \partial_{q_1} A_2 - \partial_{q_2} A_1$$

- Defines a topological invariant, the Chern number:

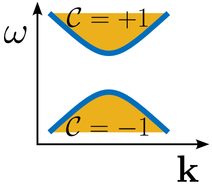
$$\mathcal{C} = \oiint F(q_1, q_2) \mathrm{d}q_1 \mathrm{d}q_2$$

- $n \neq 0$ only if **reciprocity is broken**

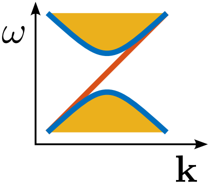


- Bulk-bondary correspondance
- Topological: **Unidirectional** edge modes

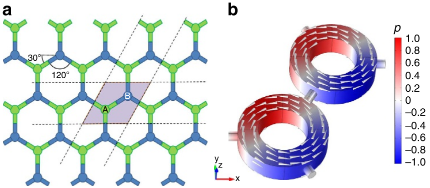
Bulk topology

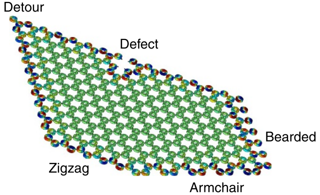


Edge modes



- Bulk-bondary correspondance
- Topological: **Unidirectional** edge modes
- Realization: flow or momentum bias





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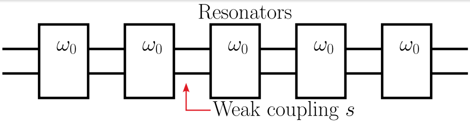
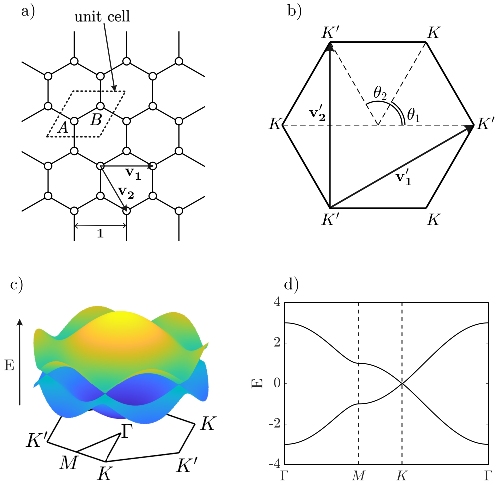


- Weakly coupled resonators
- Near a resonance frequency ω_0

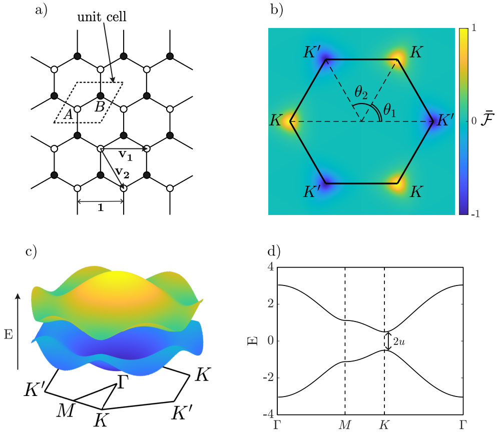
$$\omega A = \mathcal{H} A$$

- $A = (a_n)_{n \in \mathbb{Z}}$ the set of mode amplitude in each resonator
- H approximate Helmholtz equation

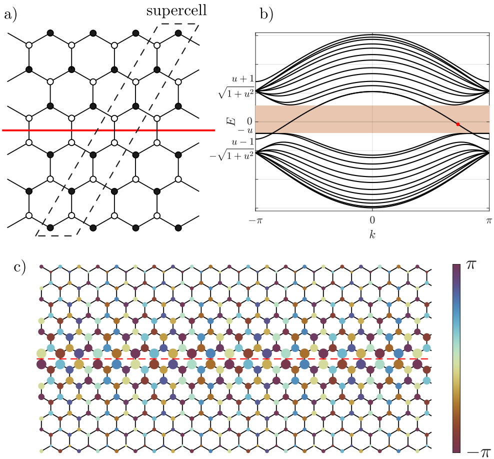
$$\mathcal{H} = \begin{pmatrix} \omega_0 & s & & \\ s & \ddots & \ddots & \\ & \ddots & \ddots & s \\ & & s & \omega_0 \end{pmatrix}$$



- Weakly coupled resonators
- Honeycomb lattice



- 2 detuned resonators (per cell)
- Opens gap
- Berry curvature has peaks



- Topological interface
- One mode per Dirac point

That's all folks

Some references:

- Periodic media:
  - Historical, quantum mechanics context:

[Ashcroft, Mermin *"Solid state physics"* (1976)]

  - Acoustic and elastic waves:

[Deymier *"Acoustic metamaterials and phononic crystals"* (2013)]

  - Transfer matrix approach (in 1D):

[Soukoulis *"Wave propagation: from electrons to photonic crystals and left-handed materials"* (2008)]

- Topological waves:

[Asboth, Oroszlany, Palyi "A short course on topological insulators" (2016)]
[Dalibard "La matière topologique et son exploration avec les gaz quantiques" (2017)]