Introduction to topological acoustics

Part II

14-16 December 2020

Master Wave Physics & Acoustics

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Previously

Bloch-Floquet method

- Periodic systems
- Bloch condition

$$\varphi_q(x+a) = e^{iq}\varphi_q(x)$$

- Bloch eigen-value problem

$$\varepsilon(q)\varphi_q = H(q) \cdot \varphi_q$$

- $\varepsilon(q)$ gives energy bands ($\sim$ frequency bands)

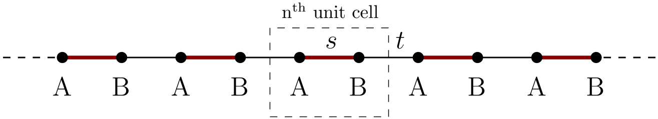
The SSH model

- Two bands
- Finite systems: **edge effects?**

Outline

Outline

Infinite SSH model

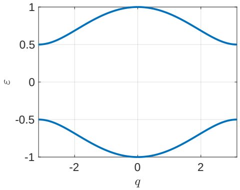


- Full eigen-value problem $\varepsilon\phi = \hat{H} \cdot \phi$

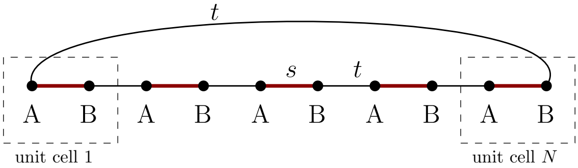
$$
\begin{aligned}
\varepsilon\phi_n^A &= t\phi_{n-1}^B + s\phi_n^B \\
\varepsilon\phi_n^B &= t\phi_{n+1}^A + s\phi_n^A
\end{aligned}
$$

- Bloch eigen-value problem

$$
\varepsilon \begin{pmatrix} \varphi_A \\ \varphi_B \end{pmatrix} = \begin{pmatrix} 0 & s + te^{-iq} \\ s + te^{iq} & 0 \end{pmatrix} \cdot \begin{pmatrix} \varphi_A \\ \varphi_B \end{pmatrix}
$$

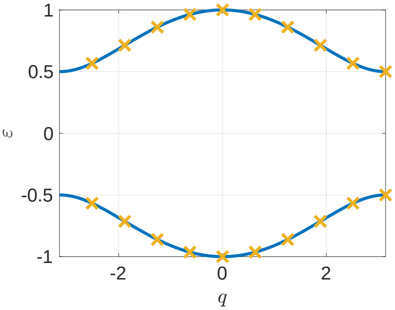


Finite SSH model on a circle

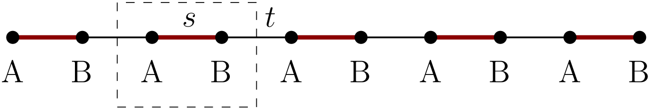


- Unit cell 1 and $N+1$ are identical

$$e^{i(N+1)q} = 1 \quad \Rightarrow \quad q = \frac{2j\pi}{N+1}$$

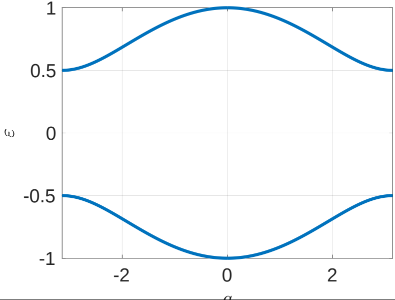


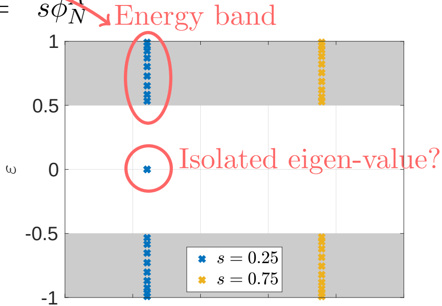
Finite SSH model



- At edges

$$\varepsilon\phi_1^A = \quad s\phi_1^B$$
$$\varepsilon\phi_N^B = \quad s\phi_N^A$$





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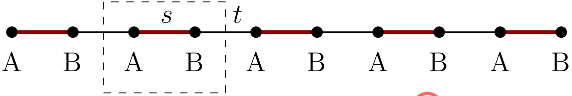
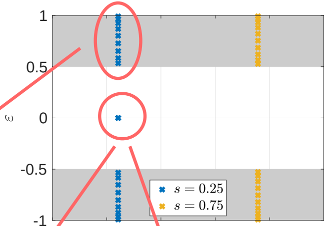
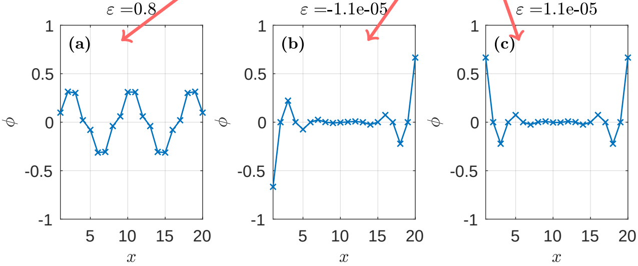


- Eigen-value problem

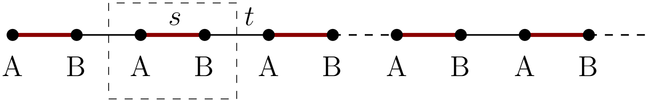
$$\varepsilon \phi = H \cdot \phi$$

- Edge mode for $\varepsilon \approx 0$





Semi-infinite chain



- Eigen-value problem for $\varepsilon = 0$

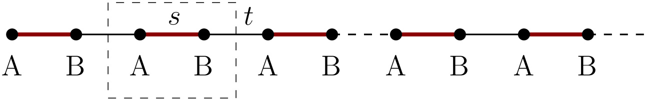
$$\begin{aligned} 0 &= s\phi_n^B + t\phi_{n-1}^B, \qquad (2 \leq n), \\ 0 &= s\phi_n^A + t\phi_{n+1}^A, \qquad (1 \leq n), \end{aligned}$$

- Solution

$$\begin{pmatrix} \phi_n^A \\ \phi_n^B \end{pmatrix} = \begin{pmatrix} a(-s/t)^n \\ b(-t/s)^n \end{pmatrix}$$

- Boundary condition $0 = s_1\phi_n^B$ and $\phi_n \underset{n\to\infty}{\to} 0$
- Normalization $\sum_n |\phi_n^A|^2 + |\phi_n^B|^2 = 1$

Semi-infinite chain

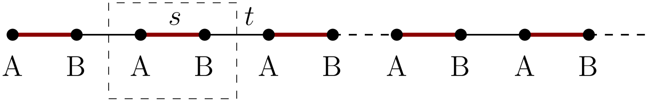


- Edge mode:

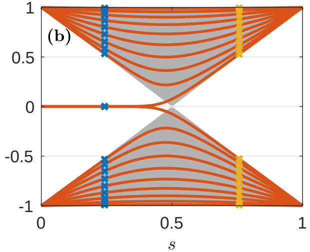
$$\psi_L = \begin{pmatrix} \phi_n^A \\ \phi_n^B \end{pmatrix} = \frac{(-s/t)^n}{\sqrt{1 - s^2/t^2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- Localized if $|s| < |t|$

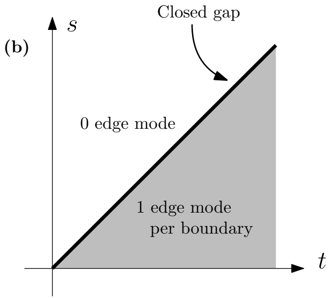
Semi-infinite chain



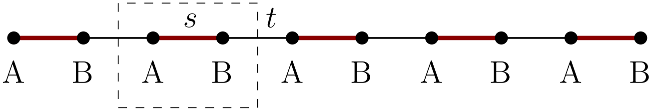
- Conclusion: edge mode localized if $|s| < |t|$
- Finite chain: 1 edge mode per boundary
- Finite spectrum for continuously changing s and t



with $s + t = 1$



Finite chain

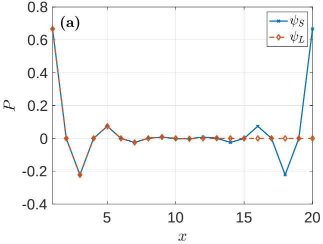


- Recall edge mode for semi-infinite chain:

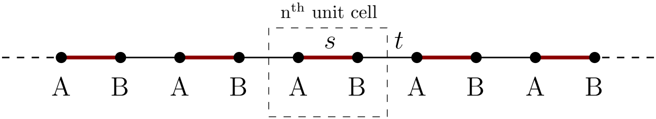
$$\psi_L = \begin{pmatrix} \phi_n^A \\ \phi_n^B \end{pmatrix} = \frac{(-s/t)^n}{\sqrt{1 - s^2/t^2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- Finite chain: 2 boundaries
- Edge modes

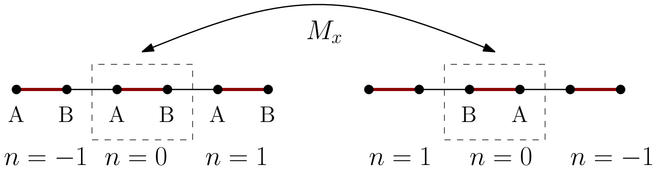
$$\psi = \frac{1}{\sqrt{2}} (\psi_L \pm \psi_R)$$



Outline



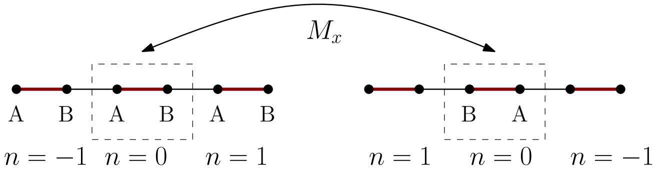
- Symmetries play an important role for topological modes
- What symmetry do we have (in addition to periodicity)?



- A classic symmetry: mirror symmetry M_x
- If center is center of cell $n = 0$

$$M_x \cdot \begin{pmatrix} \phi_n^A \\ \phi_n^B \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \phi_{-n}^A \\ \phi_{-n}^B \end{pmatrix}$$

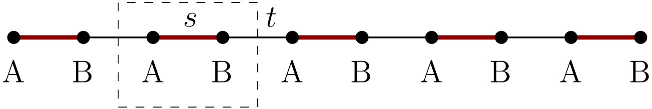
- Symmetry means $\hat{H} \cdot M_x = M_x \cdot \hat{H}$
- Can look for eigen-vectors as **symmetric** and **anti-symmetric**



- **But** M_x doesn't commute with translations
- For Bloch Hamiltonian

$$M_x \cdot H(q) \cdot M_x = H(-q)$$

- q changes as well
  - Not so convenient
  - **Still useful! (see later)**

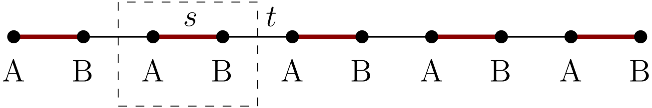


- An "exotic" symmetry: **chiral symmetry**
- Chiral operator

$$\Gamma \cdot \begin{pmatrix} \phi_n^A \\ \phi_n^B \end{pmatrix} = \begin{pmatrix} \phi_n^A \\ -\phi_n^B \end{pmatrix}$$

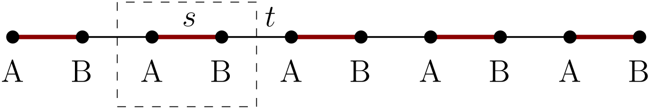
- Anti-commutation relation

$$\hat{H} \cdot \Gamma + \Gamma \cdot \hat{H} = 0$$



- **Interpretation** of chiral symmetry: Sublattice symmetry
- 2 sublattices:
  - A-points
  - B-points
- Hamiltonian contains only **cross-terms**
- Eigen-value problem of the form

$$\varepsilon[]_A = []_B$$
$$\varepsilon[]_B = []_A$$

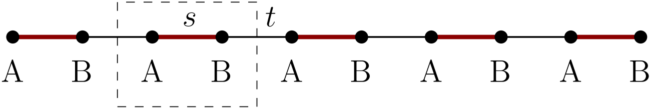


Consequences of chiral symmetry

- If (ε, ϕ) is an eigen-couple, then $(-\varepsilon, \Gamma \cdot \phi)$ also
- Spectrum symmetric $\varepsilon \to -\varepsilon$

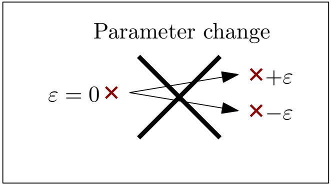
Also

- If $\varepsilon = 0$ then $\Gamma \cdot \phi = \pm\phi$
- This means $\phi_n^A = 0$ or $\phi_n^B = 0$ $(\forall n)$
- **This applies to the edge modes!**

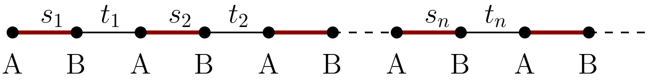


Most important consequence of chiral symmetry

- Energy of edge mode **locked** at $\varepsilon = 0$
- Single mode $\varepsilon = 0$ cannot split into a **pair**
- Valid for $N_{\text{cell}} \gg 1$



Outline



- Eigen-value problem for $\varepsilon = 0$

$$0 = s_n \phi_n^B + t_{n-1} \phi_{n-1}^B$$
$$0 = s_n \phi_n^A + t_n \phi_{n+1}^A$$

- Same derivation as before

$$\psi_L = \mathcal{N} \prod_{j=1}^{n} (-s_j/t_j) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

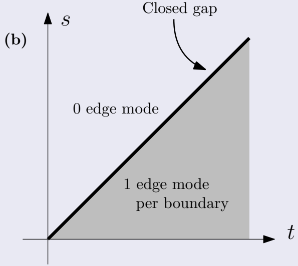
- $\mathcal{N}$ normalization constant
- **Localized if**

$$\lim_{n\to\infty} \prod_{j=1}^{n} (s_j/t_j) = 0$$

Recap'

Edge mode in SSH model

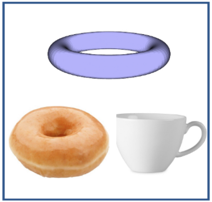
- Presence if $s < t$; absence if $s > t$
- To get rid of it: **must close gap**
- **Locked** at $\varepsilon = 0$ (chiral symmetry)
- Robust to (chiral) disorder:
  - Still localized at edge
  - Still $\varepsilon = 0$



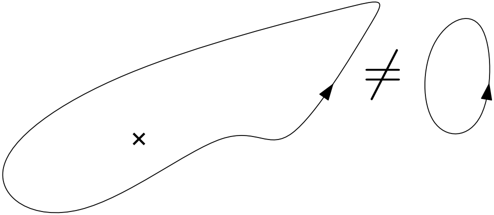
Origin of these exotic properties: topology

Outline

Topology



Study of smooth deformation



Outline

- Back to Bloch eigen-value problem

$$\varepsilon(\mathbf{q})\varphi(\mathbf{q}) = H(\mathbf{q}) \cdot \varphi(\mathbf{q})$$

- Up to now we had $\mathbf{q} = q \in]-\pi, \pi]$ with $-\pi$ and π identified
- Generalize to $\mathbf{q} \in \mathcal{Q}$ parameter space (can have d dimensions)

Uniqueness and regularity of $\varepsilon(\mathbf{q})$ and $\varphi(\mathbf{q})$?

- Assume **single** eigen-values for $H(\mathbf{q})$ for all $\mathbf{q}$:
  $\Rightarrow \varepsilon(\mathbf{q})$ unique and smooth functions
- **What about $\varphi(\mathbf{q})$?**

- Back to Bloch eigen-value problem

$$\varepsilon(\mathbf{q})\varphi(\mathbf{q}) = H(\mathbf{q}) \cdot \varphi(\mathbf{q})$$

- Ambiguity: if $\alpha(\mathbf{q})$ smooth function

$$\text{(G):} \qquad \tilde{\varphi}(\mathbf{q}) = e^{i\alpha(\mathbf{q})}\varphi(\mathbf{q})$$

- $\tilde{\varphi}(\mathbf{q})$ is equally good as $\varphi(\mathbf{q})$ (for a given eigen-value)
- We call (G) a **gauge transformation**
- Consequences of (G):
  - Function $\varphi(\mathbf{q})$ not unique
  - Not guaranteed to be smooth

Example: back to SSH

- Bloch Hamiltonian

$$H(\mathbf{q}) = \begin{pmatrix} 0 & s + te^{-iq} \\ s + te^{-iq} & 0 \end{pmatrix}$$

- Eigenvalues $\varepsilon_{\pm}(q) = \pm|s + te^{iq}| = \pm\sqrt{s^2 + t^2 + 2st\cos(q)}$
- We pick the eigen-value $\varepsilon(q) = |s + te^{iq}|$
- Eigen-vector $\varphi(q) = \begin{pmatrix} x \\ y \end{pmatrix}$ verifies

$$\varepsilon x = (s + te^{-iq})y$$

- Possible choices

$$\varphi(q) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{s+te^{-iq}}{|s+te^{iq}|} \\ 1 \end{pmatrix} \qquad \tilde{\varphi}(q) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \frac{s+te^{iq}}{|s+te^{iq}|} \end{pmatrix}$$

Example: back to SSH

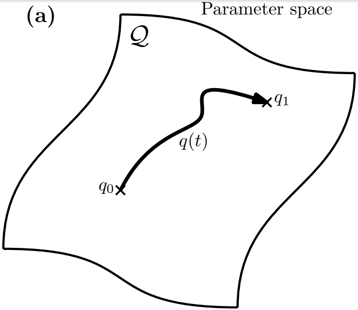
- Possible choices

$$\varphi(q) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{s+te^{-iq}}{|s+te^{iq}|} \\ 1 \end{pmatrix} \qquad \tilde{\varphi}(q) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \frac{s+te^{iq}}{|s+te^{iq}|} \end{pmatrix}$$

Is there a "natural" way of choosing the phase?

Short answer: yes and no

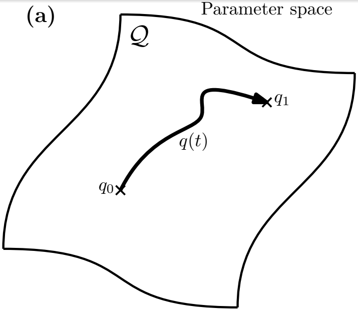
- Globally: no
- From one point to the next: yes



- Pick a given eigen-value $\varepsilon(\mathbf{q})$ (assume no degeneracy)
- Choose an arbitrary choice of eigen-vectors $\varphi(\mathbf{q})$
- Take a starting point $\mathbf{q_0} \in \mathcal{Q}$
- Take a **curve** $\mathbf{q}(t) : \mathbf{q_0} \to \mathbf{q_1}$
- Let's move along $\mathbf{q}(\mathbf{t})$ **dynamically**

Must be smooth on whole curve

$$\phi(t_0) = \varphi(q_0) \qquad \phi(t_1) = ?$$



- Dynamics as slowly as possible → **adiabatic limit**

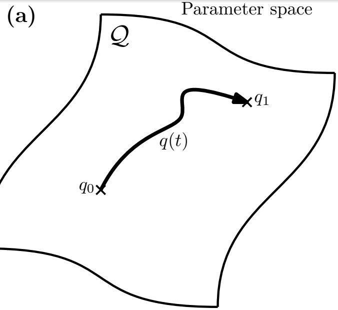
$$\phi(t) = e^{i\int A(q)\mathrm{d}q} e^{-i\int \varepsilon(q(t))\mathrm{d}t} \varphi(q(t))$$

Geometrical phase   Dynamical phase

- With

$$A(\mathbf{q}) = -i\langle\varphi|\nabla_{\mathbf{q}}\varphi\rangle$$

- This is called **parallel transport**

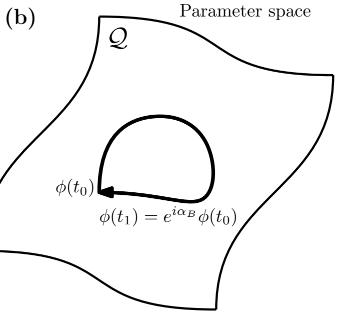


- $A(\mathbf{q})$ is called **Berry connection**

$$A(\mathbf{q}) = -i\langle\varphi|\nabla_{\mathbf{q}}\varphi\rangle$$

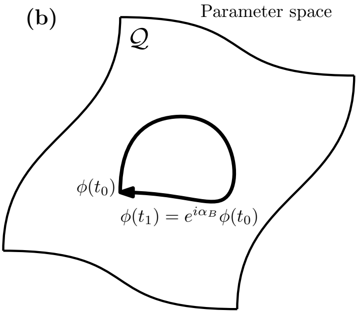
- Main properties:
  - $A(\mathbf{q})$ is real valued
  - Gauge transformation

$$\tilde{\varphi}(\mathbf{q}) = e^{i\xi(\mathbf{q})}\varphi(\mathbf{q}) \qquad \Rightarrow \qquad \tilde{A}(\mathbf{q}) = A(\mathbf{q}) + \nabla_{\mathbf{q}}\xi$$



- Important case: **curve** is a **loop**
- In general $\phi(t_1) \neq \phi(t_0)$ → **phase shift**
- This is called the Berry phase

$$\alpha_B = \oint A(q)\mathrm{d}q$$



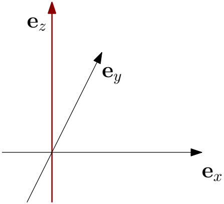
- Berry phase

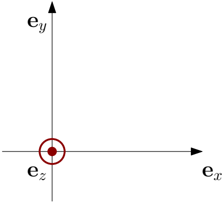
$$\alpha_B = \oint A(q)\mathrm{d}q$$

- Main properties
  - Purely geometrical: only depends on trajectory
  - Gauge independent

Outline

Example





- Real valued 2×2 matrices

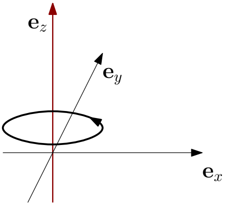
$$H = \begin{pmatrix} z + x & -y \\ -y & z - x \end{pmatrix}$$

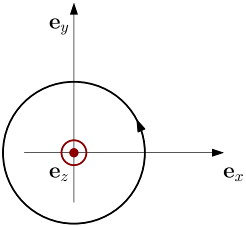
- Parameter space $\mathbb{R}^3 = \{(x, y, z)\}$
- Eigen-values

$$\varepsilon_\pm = z \pm \sqrt{x^2 + y^2}$$

- Degeneracy: z-axis

Example



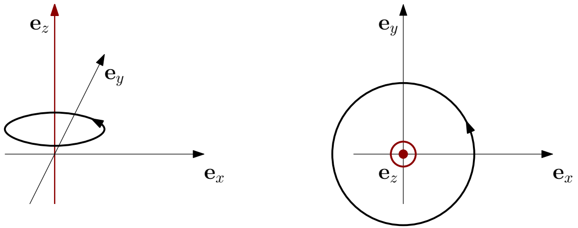


- Eigen-vectors can be chosen **real-valued**
- This implies

$$A = -i\langle\varphi|\nabla\varphi\rangle = \mathrm{Im}\left(\langle\varphi|\nabla\varphi\rangle\right) = 0$$

- Hence, phase is **always** parallel transported
  $\rightarrow \varphi(x, y, z)$ not necessarily smooth

Example



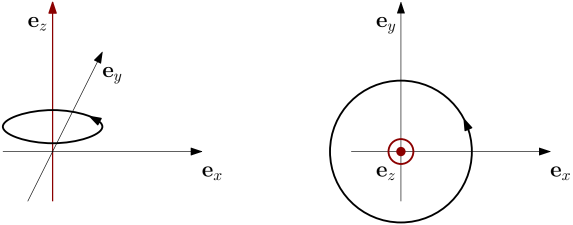
- Eigen-vector

$$\varphi_-(x,y,z) = \frac{1}{\sqrt{y^2 + (x + \sqrt{x^2+y^2})^2}} \begin{pmatrix} y \\ x + \sqrt{x^2+y^2} \end{pmatrix}$$

- Real valued but **not** smooth!

$$\lim_{\substack{x<0 \\ y \to 0^\pm}} \varphi_-(x,y,z) = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} \quad \text{sign}(y)$$

Example

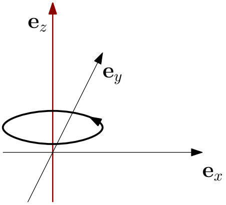


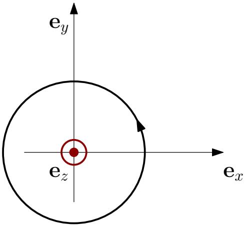
- Easier to see in cylindrical coordinates $(x, y, z) \to (r, \theta, z)$

$$\varphi_- = \begin{pmatrix} \sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}$$

- Should be $\theta \to \theta + 2\pi$ periodic!
- Jump $\lim_{\theta \to \pi} \varphi_- = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ but $\lim_{\theta \to -\pi} \varphi_- = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

Example





- Solution to get rid of discontinuity → **complex** values

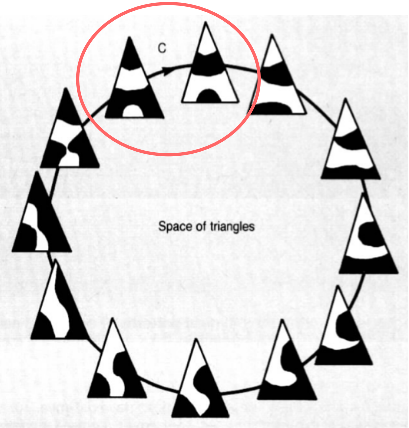
$$\varphi_- = e^{i\theta/2} \begin{pmatrix} \sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix} = \frac{i}{2} \begin{pmatrix} 1 - e^{i\theta} \\ -i(1 + e^{i\theta}) \end{pmatrix}$$

- But Berry connection is non-zero

$$e^{i \oint A d\theta} = -1$$

$$A = \frac{1}{2r} \mathbf{e}_\theta \qquad \Rightarrow \qquad \oint A \mathrm{d}\theta = \frac{1}{2} \oint \mathrm{d}\theta = \pi$$

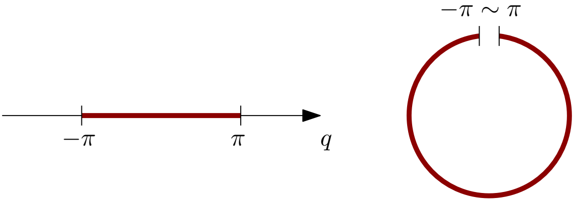
Example II



- Eigen-modes of a triangle (A, B, C) cavity
- Point A moves around a loop $\mathcal{C}$
- After a rond trip: factor -1

Outline

Zak phase



- In periodic systems: parameter is Bloch wavenumber q
- In 1D: q-space $\mathcal{Q}$ is a loop
- Each band has a Berry phase: the **Zak phase**

$$\alpha_Z = \oint_{-\pi}^{\pi} A_n(q) \mathrm{d}q$$

Zak phase: Mirror symmetric systems

- Mirror symmetry

$$M_x \cdot H(q) = H(-q) \cdot M_x$$

- If $\varphi(q)$ is eigen-vector of $H(q)$, then $M_x \cdot \varphi(q)$ is eigen-vector of $H(-q)$ with the same eigen-value
- $M_x \cdot \varphi(q)$ and $\varphi(-q)$ differ of a phase

$$\varphi(-q) = e^{i\theta(q)} M_x \cdot \varphi(q)$$

Zak phase: Mirror symmetric systems

- Mirror relation

$$(\mathrm{M}) \qquad \varphi(-q) = e^{i\theta(q)} M_x \cdot \varphi(q)$$

- Using (M) the Zak phase is

$$\alpha_Z = \theta(\pi) - \theta(0)$$

- Using (M) twice

$$e^{i\theta(q)} e^{i\theta(-q)} = 1$$

- In particular $\theta(0)$ and $\theta(\pi)$ can **only be multiples of π**

α_Z/π is an integer $\rightarrow$ **topological invariant**

Conclusion for mirror symmetric systems

- Zak phase
$$\alpha_Z = \oint_{-\pi}^{\pi} A_n(q)\mathrm{d}q$$
- Independent of choice of eigen-vectors: intrinsic property of an energy band
- α_Z/π is an **integer**
- α_Z is constant under continuous deformation
- α_Z can change only by closing the gap

Recap'

Topology

- Choice of eigen-vectors $\varphi(q)$ defines a connection

$$A(q) = -i\langle\varphi|\partial_q\varphi\rangle$$

- A leads to accumulated phase around a loop

$$\alpha = \oint A(q)\mathrm{d}q$$

Mirror symmetric systems

- Accumulated phase around Brillouin zone: **Zak phase**
- α_Z can only be multiple of π

Outline

Principle: Bulk-Boundary correspondance

$H(q)$

Infinite system: Bulk

$\hat{H}$

Finite system: Bulk + Boundary

Topological invariant ⟷ Edge modes

Correspondance

Same number $n \in \mathbb{Z}$

Topological invariant = Number of edge modes

Topological invariant = Number of edge modes

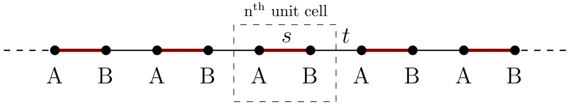
Explains:

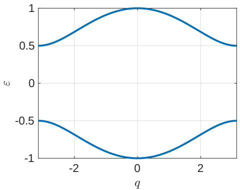
- Presence or absence of edge modes
- Mode number changes **only** if gap closes
- Frequency locking **if** combined with symmetries

Why is it useful?

- Predict the number of edge modes from **bulk properties**
- Guarantees their robustness
- Allows identification of main properties

Zak phase in SSH: direct computation





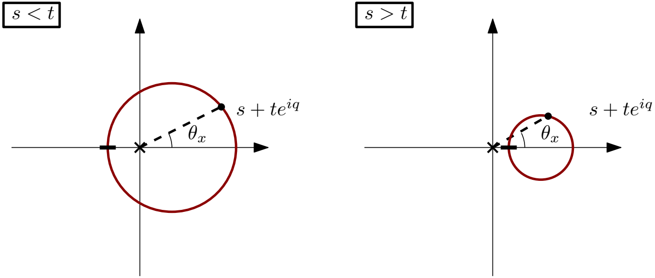
- Eigen-vector for $\varepsilon(q) = |s + te^{iq}|$ (1 possible choice)

$$\varphi(q) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\theta_x(q)} \end{pmatrix}$$

- We defined $\theta_x(q) = \arg(s + te^{iq})$ in $]-\pi, \pi]$
- Berry connection

$$A(q) = \frac{1}{2} \partial_q \theta_x$$

Zak phase in SSH: direct computation

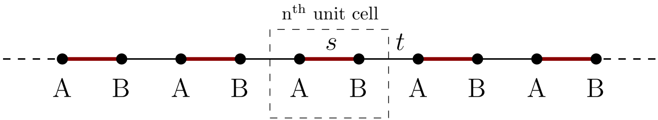


- Zak phase

$$\alpha_Z = \oint A(q)\mathrm{d}q = \frac{1}{2}\int_{-\pi}^{\pi} \partial_q \theta_x \mathrm{d}q = \frac{1}{2}(\theta_x(\pi) - \theta_x(-\pi))$$

- Two cases
  - $s < t$: $\alpha_Z = \pi$ $\Rightarrow$ **1 edge mode!**
  - $s > t$: $\alpha_Z = 0$ $\Rightarrow$ **0 edge mode!**

Zak phase in SSH: alternative method

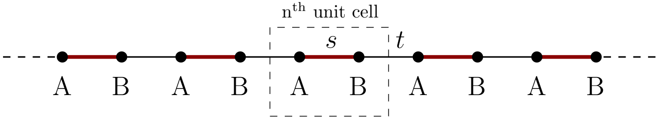


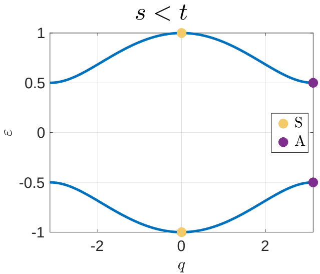
- At band edges ($q = 0$ or $q = \pm\pi$)
  eigen-vector is mirror symmetric or mirror anti-symmetric

$$
\begin{aligned}
M_x \cdot \varphi(0) &= \pm\varphi(0) \\
M_x \cdot \varphi(\pi) &= \pm\varphi(\pi)
\end{aligned}
$$

- Look at these symmetries to obtain α_Z

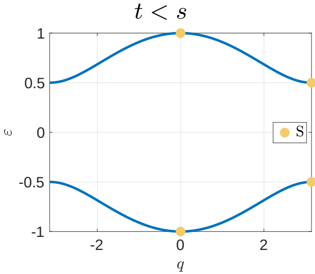
Zak phase in SSH: alternative method





S: Symmetric

A: Anti-symmetric



- $s < t$: Symmetry change $\Rightarrow \alpha_Z = \pi$ $\Rightarrow$ **1 edge mode!**
- $t < s$: No symmetry change $\Rightarrow \alpha_Z = 0$ $\Rightarrow$ **0 edge mode!**

Recap’

Bulk-boundary correspondance

$H(q)$

Infinite system: Bulk

↓

Topological invariant

$\hat{H}$

Finite system: Bulk + Boundary

↓

Edge modes

Topological invariant ⟷ Correspondance ⟷ Edge modes

Same number $n \in \mathbb{Z}$

That's all folks

Some references:

- Reviews with treatment of SSH model:

  [Dalibard "La matière topologique et son exploration avec les gaz quantiques" (2017)]

  [Asboth, Oroszlany, Palyi "A short course on topological insulators" (2016)]

- Berry phase and topology

  [Berry "Quantum adiabatic anholonomy" (1990)]

  [Budich, Trauzettel "From the adiabatic theorem of quantum mechanics to topological states of matter" (2013)]