Introduction to topological acoustics

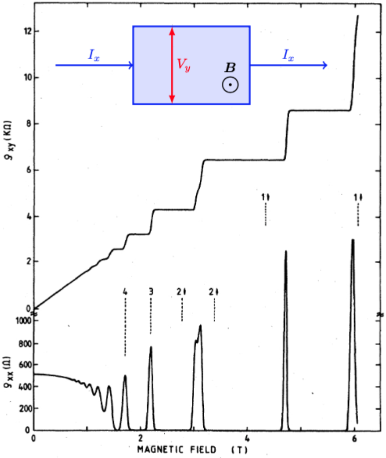
Part I

14-16 December 2020

Master Wave Physics & Acoustics

by Antonin Coutant

Brief history



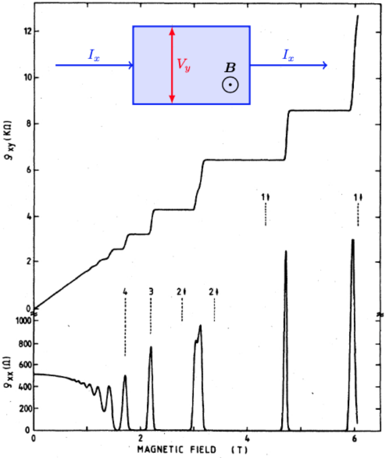
First discovery

- Quantum Hall Effect (1986)
- 2D metal in a strong magnetic field
- Resistivity $\propto 1/n$

Main theory

- Current carried by **surface waves**
- Integer number of modes → **topological invariant**

Brief history



In the '00

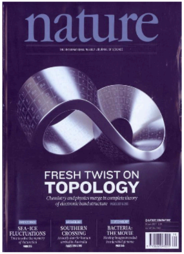
- Quantum Spin Hall Effect
- Same effect but:
  - No magnetic field
  - Uses electron spin
  - Also works in 3D

Since then:

Applications in many fields:

Topology $\Rightarrow$ surface waves with exotic properties

- Photonics
- Acoustics
- Mechanical waves
- Cold atoms



In these lectures:

- Not following history
- Constructive approach:
  Simple models $\rightarrow$ main ingredients and tools
- Focus on acoustics and mechanics

Approximate plan:

1. Periodic system: Bloch-Floquet + examples
2. Finite systems: surface waves and properties
3. Tools of topology
4. Focus on 1D and then a 2D case

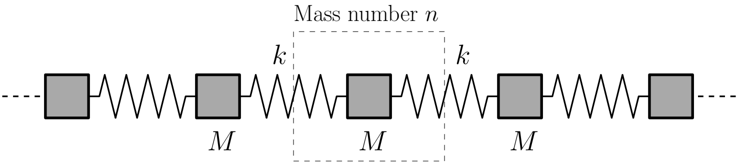
Outline

1. Bloch-Floquet formalism
   - A first example
   - General method
2. Application to examples
   - Example I: masses and springs
   - Example II: waveguide
3. The SSH model
   - Bloch modes
   - Boundary effects

Outline

1. Bloch-Floquet formalism
   - A first example
   - General method
2. Application to examples
   - Example I: masses and springs
   - Example II: waveguide
3. The SSH model
   - Bloch modes
   - Boundary effects

First example: Chain of masses and springs

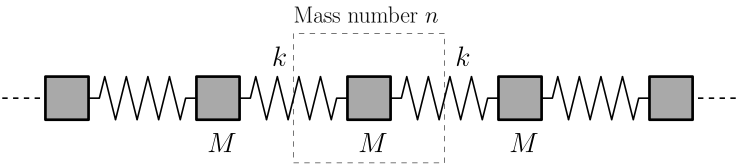


- Label each mass by an integer $n \in \mathbb{Z}$
- Displacement from equilibrium: $X_n(t)$
- Newton's second law:

$$M\ddot{X}_n = -k(X_n - X_{n-1}) - k(X_n - X_{n+1})$$

- Solutions?

First example: Chain of masses and springs

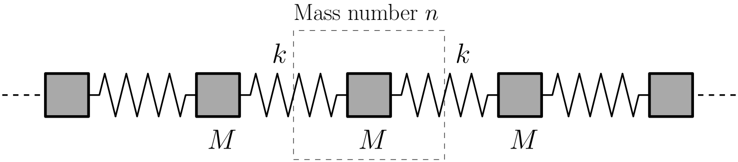


- Excitation of a single mass at frequency f?
- Plane wave like:

$$X_n(t) = \mathrm{Re}\left(\underline{\mathrm{x}}\, e^{-i\omega t + inq}\right)$$

- With
  - $\omega = 2\pi f$: angular frequency
  - q: quasi-wavenumber
  - $\underline{\mathrm{x}}$: complex amplitude

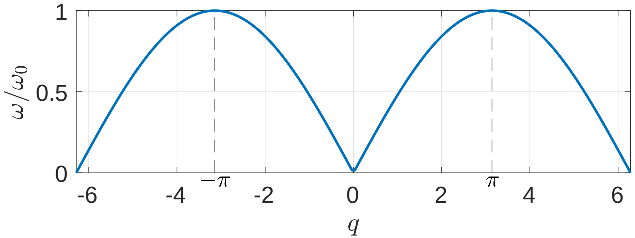
First example: Chain of masses and springs



- Using $X_n(t) = \mathrm{Re}\left(\underline{\mathrm{x}}\, e^{-i\omega t + inq}\right)$ in

$$M\ddot{X}_n = -k(X_n - X_{n-1}) - k(X_n - X_{n+1})$$

First example: Chain of masses and springs



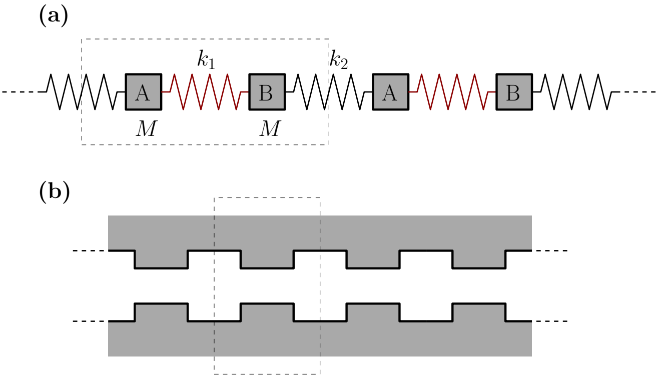
- Dispersion relation:

$$\omega^2 = \frac{4k}{M}\sin^2(q/2)$$

Take away message:

- Solutions are plane wave like
- Range of allowed frequency: **passing band**
- Dispersive waves
- Periodicity in reciprocal space $q \to q + 2\pi$

General method for periodic systems?



We consider two classes of systems:

(a) Discrete: chain of equal masses and alternating springs

(b) Continuous: air channel with changing cross-section

Outline

1. Bloch-Floquet formalism
   - A first example
   - General method
2. Application to examples
   - Example I: masses and springs
   - Example II: waveguide
3. The SSH model
   - Bloch modes
   - Boundary effects

General framework

- Eigen-value equation

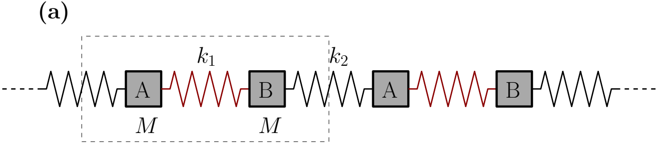
$$\varepsilon\phi = \hat{H} \cdot \phi$$

- Quantum mechanics analogy
  - Linear operator $\hat{H}$ is called Hamiltonian
  - ε is called energy (usually related to frequency)
- Vectors: ϕ function of $x \in \mathcal{M}$
- $\mathcal{M}$ is configuration space, it can be
  - (a) **Discrete:** $\mathcal{M} = \mathbb{Z}$
  - (b) **Continuous:** $\mathcal{M} = \mathbb{R}$
- More generally $\mathcal{M} = \mathbb{Z} \times \mathcal{I}$ or $\mathcal{M} = \mathbb{R} \times \mathcal{I}$

  $\mathbb{Z}$ or $\mathbb{R}$: relevant direction with periodicity

  $\mathcal{I}$: degrees of freedom not translated (internal, other directions)

Examples



- Two displacements $X_n^A(t)$ and $X_n^B(t)$: $\mathcal{M} = \mathbb{Z} \times \mathcal{I}$
- Fixed frequency $X_n^{A/B}(t) = \text{Re}\left(x_n^{A/B} e^{-i\omega t}\right)$
- Newton's second law

$$\begin{aligned} \omega^2 x_n^A &= \frac{k_1}{M}(x_n^A - x_n^B) + \frac{k_2}{M}(x_n^A - x_{n-1}^B), \\ \omega^2 x_n^B &= \frac{k_1}{M}(x_n^B - x_n^A) + \frac{k_2}{M}(x_n^B - x_{n+1}^A). \end{aligned}$$

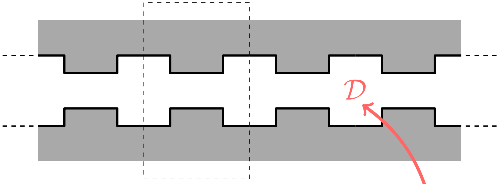
- Hence,

$$\varepsilon = \omega^2$$

$$\hat{H} \cdot \phi = \text{Right-hand side}$$

Examples

(b)



- Pressure $p(x,y)$: $\mathcal{M} = \mathbb{R} \times \mathcal{I}$ $(y \in \mathcal{I})$
- Fixed angular frequency $\omega = kc_0$
- Helmholtz equation

$$\begin{aligned} k^2 p + \Delta p &= 0 \quad \text{in} \quad \mathcal{D} \\ \partial_n p &= 0 \quad \text{on} \quad \partial\mathcal{D} \end{aligned}$$

- Hence,

$$\varepsilon = k^2$$

$$\hat{H} = -\Delta_{\mathcal{D}}$$

Assumption:

- We assume $\hat{H}$ is **self-adjoint** (or **hermitian**)
- Define scalar product

$$\begin{aligned} \langle \phi_1 | \phi_2 \rangle &= \sum_{\mathcal{M}} \phi_1(x)^* \phi_2(x) \qquad \text{(discrete)} \\ \langle \phi_1 | \phi_2 \rangle &= \int_{x \in \mathcal{M}} \phi_1(x)^* \phi_2(x) \mathrm{d}x \qquad \text{(continuous)} \end{aligned}$$

- Self-adjoint means

$$\langle \phi_1 | \hat{H} \cdot \phi_2 \rangle = \langle \hat{H} \cdot \phi_1 | \phi_2 \rangle$$

- if $\hat{H}$ is a matrix, it means $\hat{H} = \hat{H}^{*T}$

Assumption:

- We assume $\hat{H}$ is **self-adjoint** (or **hermitian**)
- Very convenient: spectral theorem
  - Basis (ϕ_j) such that

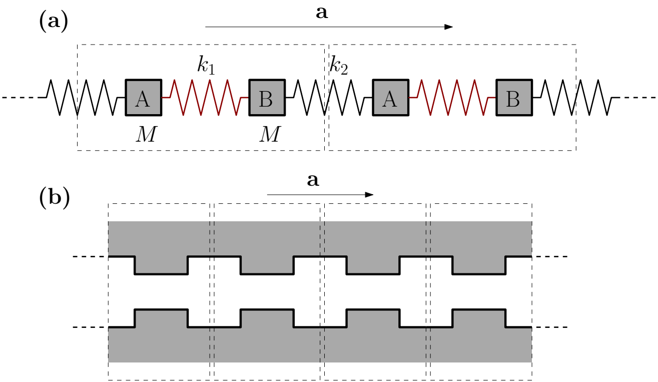
$$\varepsilon_j \phi_j = \hat{H} \cdot \phi_j$$

  - Orthonormal $\langle \phi_j | \phi_{j'} \rangle = \delta_{jj'}$
- Interpretation: usually comes from **energy conservation**
- **Careful:** has to be checked case by case

Periodic systems

- $\mathbf{a} \in \mathcal{M}$ such system is invariant under translation by $\mathbf{a}$
- Also invariant under $n \times \mathbf{a}$ (n integer) $\rightarrow$ smallest $\mathbf{a}$ optimal

Examples



General framework

- **Translation operator $T_{\mathbf{a}}$**

$$(T_{\mathbf{a}} \cdot \phi)(x) = \phi(x + \mathbf{a})$$

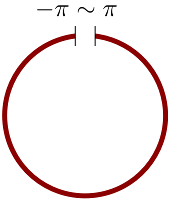
- Periodicity means

$$T_{\mathbf{a}} \cdot \hat{H} \cdot T_{\mathbf{a}}^{-1} = \hat{H}$$

- This is equivalent to $T_{\mathbf{a}} \cdot \hat{H} = \hat{H} \cdot T_{\mathbf{a}}$
  $\Rightarrow$ **co-diagonalization**

General framework

- Diagonalize $T_{\mathbf{a}}$ first
$$T_{\mathbf{a}} \cdot \phi = \lambda \phi$$
- **But** $T_{\mathbf{a}}$ conserves norm (unitary operator)
$$||T_{\mathbf{a}} \cdot \phi||^2 = \int_{\mathcal{M}} |\phi(x+a)|^2 dx = \int_{\mathcal{M}} |\phi(x)|^2 dx$$
hence, $\lambda = e^{iq}$
- Reciprocal periodicity $q \to q + 2\pi$
- Restricted to $-\pi < q \leq \pi$ (first Brillouin zone)



General framework

- **Eigen-vector** of $T_{\mathbf{a}}$ satisfies **Bloch condition**

$$\phi(x+\mathbf{a}) = e^{iq}\phi(x)$$

- Alternative form: **plane wave times periodic function**

$$\phi(x) = e^{i\kappa x}U(x)$$

with U periodic: $U(x+\mathbf{a}) = U(x)$ and $\kappa = q/a$

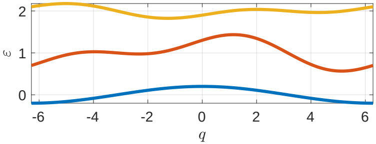
General framework

- For a given $-\pi < q \leq \pi$
- Solution of Bloch condition $\varphi(q) : x \mapsto \varphi(q;x)$
- Bloch eigen-value problem

$$H(q) \cdot \varphi(q) = \varepsilon(q)\varphi(q)$$

- Bloch Hamiltonian $H(q) = Q(q) \cdot \hat{H} \cdot Q(q)$
  $\rightarrow Q$ projector on q-space

- $H(q)$ as a discrete spectrum:
  $\varepsilon_j(q) \rightarrow$ **energy band**
  ($\sim$ frequency bands)



Bloch-Floquet in a nutshell

Bloch waves:

- System invariant under $x \to x + \mathbf{a}$
- Look for solution of the form

$$\varphi_q(x+a) = e^{iq}\varphi_q(x)$$

- $\kappa = q/a$ is Bloch wavenumber
(We call q Bloch wavenumber when no ambiguity)

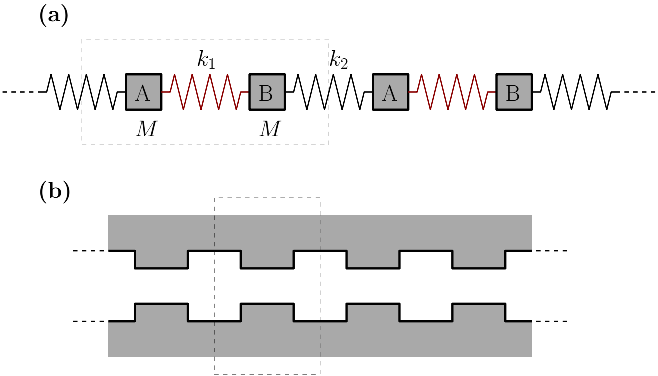
Bands and gaps:

- Eigen-value problem restricted to φ_q

$$H(q) \cdot \varphi_q = \varepsilon(q)\varphi_q$$

- Discrete eigen-values $\varepsilon_j(q) \to$ bands

Application to our 2 examples



(a) Discrete: chain of equal masses and alternating springs

(b) Continuous: air channel with changing cross-section

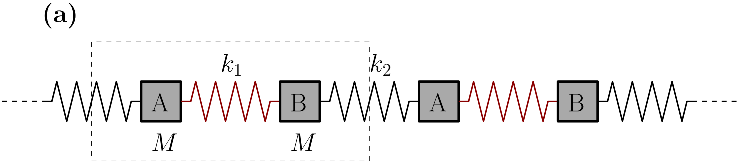
Outline

1. Bloch-Floquet formalism
   - A first example
   - General method
2. Application to examples
   - Example I: masses and springs
   - Example II: waveguide
3. The SSH model
   - Bloch modes
   - Boundary effects

Outline

1. Bloch-Floquet formalism
   - A first example
   - General method
2. Application to examples
   - Example I: masses and springs
   - Example II: waveguide
3. The SSH model
   - Bloch modes
   - Boundary effects

Example I



- Translation operator

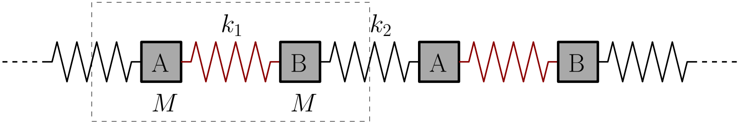
$$T_{\mathbf{a}} : \begin{cases} x_n^A \to & x_{n+1}^A \\ x_n^B \to & x_{n+1}^B \end{cases}$$

- Bloch condition solved for

$$\begin{aligned} x_n^A &= e^{inq}\varphi_A \\ x_n^B &= e^{inq}\varphi_B \end{aligned}$$

Example I

(a)



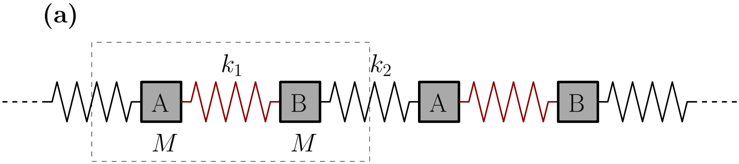
- Newton's second law

$$\omega^2 x_n^A = \frac{k_1}{M}(x_n^A - x_n^B) + \frac{k_2}{M}(x_n^A - x_{n-1}^B),$$
$$\omega^2 x_n^B = \frac{k_1}{M}(x_n^B - x_n^A) + \frac{k_2}{M}(x_n^B - x_{n+1}^A).$$

- (Recall $\varepsilon = \omega^2$.) Bloch eigen-value problem

$$\varepsilon\varphi_A = \frac{k_1}{M}(\varphi_A - \varphi_B) + \frac{k_2}{M}(\varphi_A - e^{-iq}\varphi_B),$$
$$\varepsilon\varphi_B = \frac{k_1}{M}(\varphi_B - \varphi_A) + \frac{k_2}{M}(\varphi_B - e^{iq}\varphi_A).$$

Example I

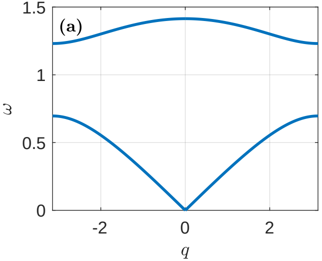


- Bloch Hamiltonian

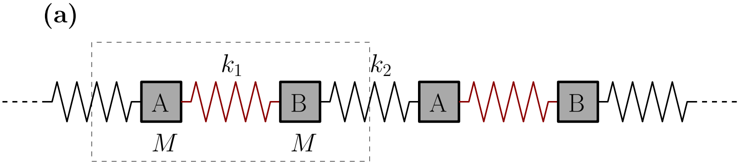
$$H(q) = \frac{1}{M} \begin{pmatrix} k_1 + k_2 & -k_1 - k_2 e^{-iq} \\ -k_1 - k_2 e^{iq} & k_1 + k_2 \end{pmatrix}$$

- Eigen-values

$$\varepsilon = \frac{k_1 + k_2}{M} \pm \frac{|k_1 + k_2 e^{iq}|}{M}$$



Example I



- Simplify the model:
  define $\omega_0 = (k_1 + k_2)/M$ (mean frequency)
- Redefine $\varepsilon = 1 - \omega^2/\omega_0^2$ (relative frequency shift)
- Then the Bloch Hamiltonian becomes

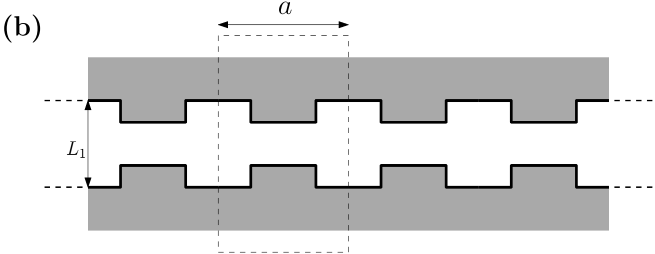
$$H(q) = \begin{pmatrix} 0 & s + te^{-iq} \\ s + te^{iq} & 0 \end{pmatrix}$$

→ **This is the Su-Schrieffer-Heeger (SSH) model**
→ **We will study this model in depth**

Outline

1. Bloch-Floquet formalism
   - A first example
   - General method
2. Application to examples
   - Example I: masses and springs
   - Example II: waveguide
3. The SSH model
   - Bloch modes
   - Boundary effects

Example II



- Translation operator

$$T_{\mathbf{a}} : p(x, y) \to p(x + a, y)$$

- Bloch condition

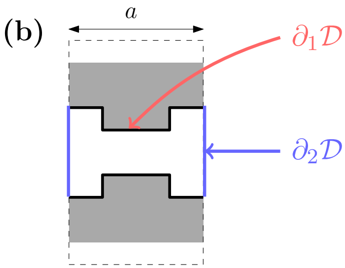
$$p(x + a, y) = e^{iq} p(x, y)$$

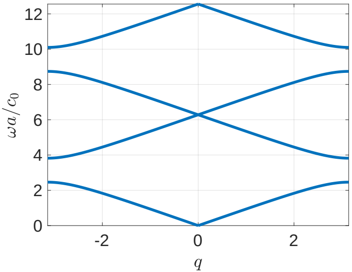
More convenient in practice

- Alternative form

$$p(x, y) = e^{iqx/a} U(x, y)$$

Example II

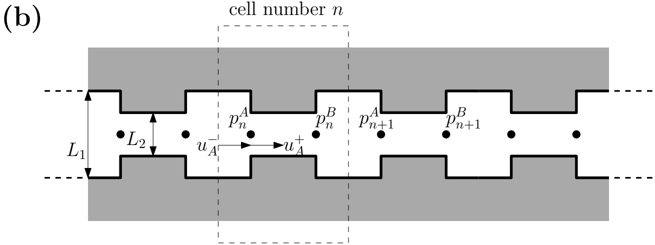




- Eigen-value problem

$$
\begin{aligned}
\varepsilon\phi &= -\Delta\phi \quad \text{on} \quad \mathcal{D} \\
\partial_n\phi &= 0 \quad \text{on} \quad \partial_1\mathcal{D} \\
\phi(x+a,y) &= e^{iq}\phi(x,y) \quad \text{on} \quad \partial_2\mathcal{D}
\end{aligned}
$$

Example II



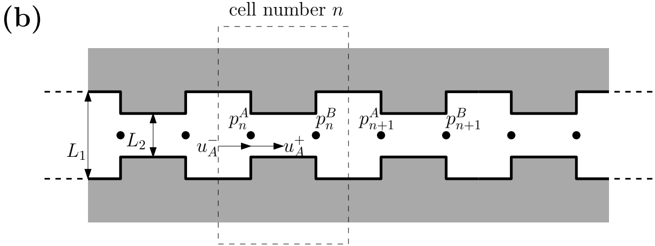
- **Simplification** for narrow tubes $L \ll a$
- Along a piece of tube (constant section)

$$p'' + k^2 p = 0$$

- Cross-section changes: p and acoustic debit continuous

$$L_1 u_A^- = L_2 u_A^+$$

Example II



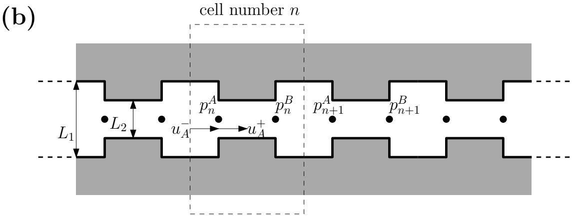
- Integrating along tubes

$$\varepsilon p_n^A = \quad t p_{n-1}^B + s p_n^B$$
$$\varepsilon p_n^B = \quad t p_{n+1}^A + s p_n^A$$

- With $\varepsilon = \cos(\omega a/2c_0)$ and

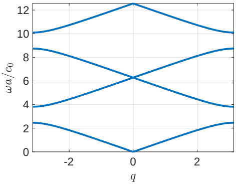
$$s = \frac{L_2}{L_1 + L_2} \qquad t = \frac{L_1}{L_1 + L_2}$$

Example II

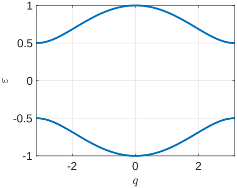


- Bloch problem → **SSH model** again

$$\varepsilon \begin{pmatrix} p_A \\ p_B \end{pmatrix} = \begin{pmatrix} 0 & s + te^{-iq} \\ s + te^{iq} & 0 \end{pmatrix} \cdot \begin{pmatrix} p_A \\ p_B \end{pmatrix}$$



→



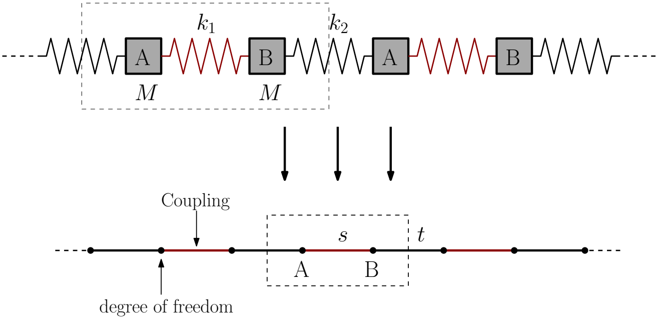
Outline

1. Bloch-Floquet formalism
   - A first example
   - General method
2. Application to examples
   - Example I: masses and springs
   - Example II: waveguide
3. The SSH model
   - Bloch modes
   - Boundary effects

Outline

1. Bloch-Floquet formalism
   - A first example
   - General method
2. Application to examples
   - Example I: masses and springs
   - Example II: waveguide
3. The SSH model
   - Bloch modes
   - Boundary effects

Lattice model

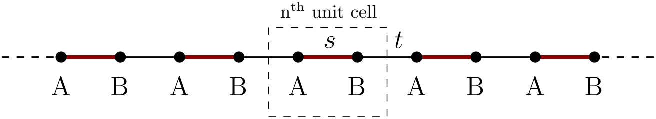


- Eigen-value problem $\varepsilon\phi = H \cdot \phi$

$\mu = A$ or B

(n, μ) —t— (n', μ'): $\langle n, \mu | H | n', \mu' \rangle = t$

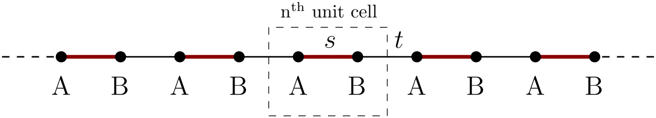
(n, μ) —s— (n', μ'): $\langle n, \mu | H | n', \mu' \rangle = s$



- Eigen-value problem $\varepsilon\phi = H \cdot \phi$

$$\begin{aligned} \varepsilon\phi_n^A &= \quad t\phi_{n-1}^B + s\phi_n^B \\ \varepsilon\phi_n^B &= \quad t\phi_{n+1}^A + s\phi_n^A \end{aligned}$$

- Assume $s > 0$ and $t > 0$

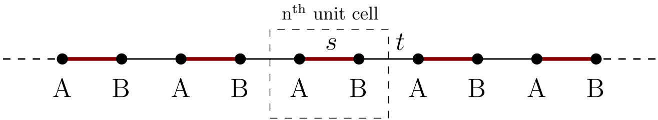


- Bloch condition

$$\begin{aligned} \phi_n^A &= e^{inq}\varphi_A \\ \phi_n^B &= e^{inq}\varphi_B \end{aligned}$$

- Bloch eigen-value problem

$$\varepsilon \begin{pmatrix} \varphi_A \\ \varphi_B \end{pmatrix} = \begin{pmatrix} 0 & s + te^{-iq} \\ s + te^{iq} & 0 \end{pmatrix} \cdot \begin{pmatrix} \varphi_A \\ \varphi_B \end{pmatrix}$$



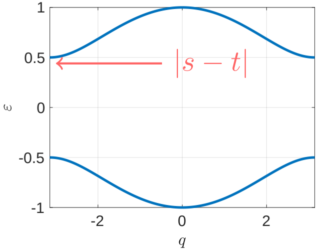
- Bloch eigen-value problem

$$\varepsilon \begin{pmatrix} \varphi_A \\ \varphi_B \end{pmatrix} = \begin{pmatrix} 0 & s + te^{-iq} \\ s + te^{iq} & 0 \end{pmatrix} \cdot \begin{pmatrix} \varphi_A \\ \varphi_B \end{pmatrix}$$

- As before: two bands and a gap

$$\varepsilon = \pm |s + te^{iq}|$$

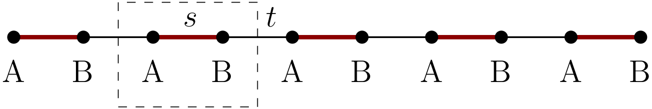
- Gap between $\pm|s - t|$



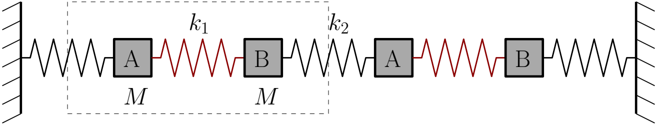
Outline

1. Bloch-Floquet formalism
   - A first example
   - General method
2. Application to examples
   - Example I: masses and springs
   - Example II: waveguide
3. The SSH model
   - Bloch modes
   - Boundary effects

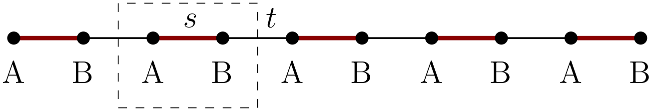
Finite chain



- Going back to masses and spring system
- Boundary conditions: last masses attached to **hard walls**



Finite chain



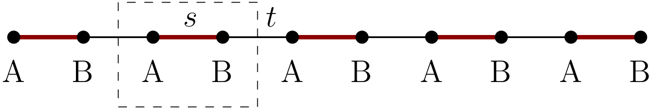
- Eigen-value problem $\varepsilon \phi = H \cdot \phi$

$$\begin{aligned} \varepsilon \phi_n^A &= s\phi_n^B + t\phi_{n-1}^B, \qquad (2 \leq n \leq N) \\ \varepsilon \phi_n^B &= s\phi_n^A + t\phi_{n+1}^A, \qquad (1 \leq n \leq N-1) \end{aligned}$$

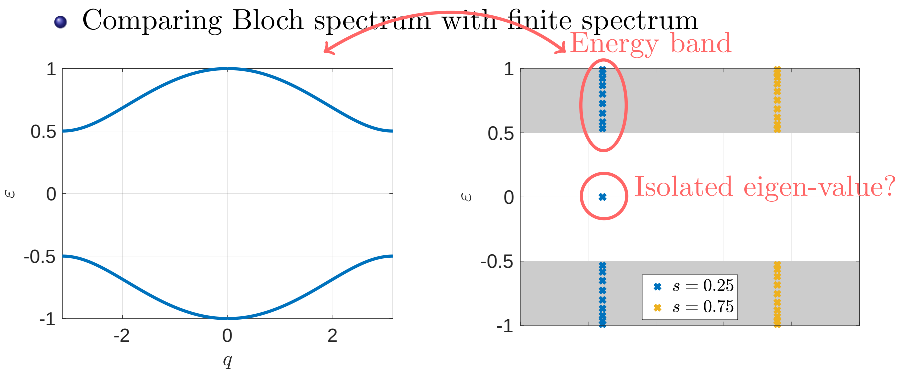
and at the edges

$$\begin{aligned} \varepsilon \phi_1^A &= s\phi_1^B \\ \varepsilon \phi_N^B &= s\phi_N^A \end{aligned}$$

Finite chain



- Comparing Bloch spectrum with finite spectrum



Recap'

Bloch-Floquet method

- Periodic systems
- Bloch condition

$$\phi(x+a) = e^{iq}\phi(x)$$

- Bloch eigen-value problem $\varepsilon(q)\varphi = H(q) \cdot \varphi$
- $\varepsilon(q)$ gives energy bands ($\sim$ frequency bands)

The SSH model

- Two bands
- Finite systems: **edge effects?**

That's all folks

Some references:

- Bloch-Floquet:
  [Deymier "Acoustic metamaterials and phononic crystals" (2013)]
  [Ashcroft, Mermin "Solid state physics" (1976)]
- Reviews with treatment of SSH model:
  [Asboth, Oroszlany, Palyi "A short course on topological insulators" (2016)]
  [Dalibard "La matière topologique et son exploration avec les gaz quantiques" (2017)]